B.Sc Part I (Hons.) Ist Paper

TRIGONOMETRY (SUMMATION OF SERIES) (Contd.)

Q. If $\theta - \alpha = \tan^2\frac{\phi}{2}\sin 2\theta - \frac{1}{2}\tan^4\frac{\phi}{2}\sin 4\theta + \frac{1}{3}\tan^6\frac{\phi}{2}\cdot\sin 6\theta - \cdots$ to ∞,

prove that $\tan\alpha = \tan\theta\cdot\cos\phi$.

Soln Given that

$$\theta - \alpha = \tan^2\frac{\phi}{2}\sin 2\theta - \frac{1}{2}\tan^4\frac{\phi}{2}\sin 4\theta + \frac{1}{3}\tan^6\frac{\phi}{2}\sin 6\theta - \cdots \text{ to } \infty$$

$= S$ (say).

Let $C = \tan^2\frac{\phi}{2}\cos 2\theta - \frac{1}{2}\tan^4\frac{\phi}{2}\cos 4\theta + \frac{1}{3}\tan^6\frac{\phi}{2}\cos 6\theta - \cdots$ to ∞

$$\therefore C + iS = \tan^2\frac{\phi}{2}(\cos 2\theta + i\sin 2\theta) - \frac{1}{2}\tan^4\frac{\phi}{2}(\cos 4\theta + i\sin 4\theta) + \frac{1}{3}\tan^6\frac{\phi}{2}(\cos 6\theta + i\sin 6\theta) - \cdots \text{ to } \infty$$

$$\Rightarrow C + iS = \tan^2\frac{\phi}{2}\cdot e^{2i\theta} - \frac{1}{2}\tan^4\frac{\phi}{2}\cdot e^{4i\theta} + \frac{1}{3}\tan^6\frac{\phi}{2}\, e^{6i\theta} - \cdots \text{ to } \infty$$

Put $\tan^2\frac{\phi}{2}\, e^{2i\theta} = x$

$$\Rightarrow C + iS = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{ to } \infty$$

Now $\boxed{\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots \text{ to } \infty}$

$$\Rightarrow c+is = \log\left(1+\tan^2\frac{\phi}{2}\cdot e^{2i\theta}\right)$$

$$\Rightarrow c+is = \log\left\{1+\tan^2\frac{\phi}{2}\cdot(\cos 2\theta + i\sin 2\theta)\right\}$$

$$\Rightarrow c+is = \log\left\{\left(1+\tan^2\frac{\phi}{2}\cos 2\theta\right) + i\tan^2\frac{\phi}{2}\sin 2\theta\right\}$$

$$\Rightarrow c+is = \frac{1}{2}\log\left\{\left(1+\tan^2\frac{\phi}{2}\cos 2\theta\right)^2 + \left(\tan^2\frac{\phi}{2}\sin 2\theta\right)^2\right\}$$

$$+ i\tan^{-1}\frac{\tan^2\frac{\phi}{2}\cdot\sin 2\theta}{1+\tan^2\frac{\phi}{2}\cdot\cos 2\theta}$$

$$\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$$

separating imaginary parts, we get

$$s = \tan^{-1}\frac{\tan^2\frac{\phi}{2}\sin 2\theta}{1+\tan^2\frac{\phi}{2}\cos 2\theta}$$

But $\theta-\alpha = s$

$$\Rightarrow \theta-\alpha = \tan^{-1}\frac{\tan^2\frac{\phi}{2}\sin 2\theta}{1+\tan^2\frac{\phi}{2}\cos 2\theta}$$

$$\Rightarrow \tan(\theta-\alpha) = \frac{\tan^2\frac{\phi}{2}\sin 2\theta}{1+\tan^2\frac{\phi}{2}\cos 2\theta}$$

$$\Rightarrow \frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)} = \frac{\tan^2\frac{\phi}{2}\sin 2\theta}{1+\tan^2\frac{\phi}{2}\cos 2\theta}$$

$$\Rightarrow \sin(\theta-\alpha)\left[1+\tan^2\frac{\phi}{2}\cos 2\theta\right] = \tan^2\frac{\phi}{2}\sin 2\theta\cos(\theta-\alpha)$$

$$\Rightarrow \sin(\theta-\alpha)+\sin(\theta-\alpha)\tan^2\frac{\phi}{2}\cos 2\theta.$$

$$= \tan^2\frac{\phi}{2}\cdot\sin 2\theta\cos(\theta-\alpha)$$

$$\Rightarrow \sin(\theta-\alpha) = \tan^2\frac{\phi}{2}\left[\sin 2\theta\cdot\cos(\theta-\alpha)-\sin(\theta-\alpha)\cos 2\theta\right]$$

$$\Rightarrow \sin(\theta-\alpha) = \tan^2\frac{\phi}{2}\cdot\sin\{2\theta-(\theta-\alpha)\}$$

$$\Rightarrow \sin(\theta-\alpha) = \tan^2\frac{\phi}{2}\cdot\sin(\theta+\alpha)$$

$$\Rightarrow \frac{1}{\tan^2\frac{\phi}{2}} = \frac{\sin(\theta+\alpha)}{\sin(\theta-\alpha)}$$

By componendo & dividendo,

$$\frac{1+\tan^2\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} = \frac{\sin(\theta+\alpha)+\sin(\theta-\alpha)}{\sin(\theta+\alpha)-\sin(\theta-\alpha)}$$

or $$\frac{1+\tan^2\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}} = \frac{\cancel{2}\sin\theta\cdot\cos\alpha}{\cancel{2}\cos\theta\cdot\sin\alpha}$$

or, $$\frac{1}{\cos\phi} = \tan\theta\cdot\cot\alpha$$

$$\Rightarrow \boxed{\tan\alpha = \tan\theta\cdot\cos\phi}$$ Proved